Remark 1°. The quantities $\delta\mathbf{u} = \partial\mathbf{u}(\mathbf{x}, 0)/\partial\eta$ and $\delta^2\mathbf{u} = \partial^2\mathbf{u}(\mathbf{x}, 0)/\partial\eta^2$ are, respectively, called the first and second variations of the displacement $\mathbf{u}$. It follows from Theorems 1 and 2 that $\delta\mathbf{u}, \delta^2\mathbf{u} \in V(\Omega^*)$.

Remark 2°. If $r(\mathbf{y}^*, \eta) < 0$ the proof of Theorem 1 will differ somewhat from that presented. In particular, the differentiability of $\mathbf{u}$ and $\mathbf{w}$ should be taken into account in $\Omega\setminus\bar{\Omega}^*$ and Ω should be replaced by Ω^+ in the integral identities (3.2), (3.5), (3.10), the estimates (3.6), (3.11), (3.12), and in the limit (3.8).

REFERENCES

1. PRAGER W., Principles of the Theory of Optimal Structural Design. Mir, Moscow, 1977.
2. TROITSKII V.A. and PETUKHOV L.V., Optimization of the Shape of Elastic Bodies, Nauka, Moscow, 1982.
3. BANICHUK N.V., BEL'SKII V.G. and KOBELEV V.V., Optimization in problems of the theory of elasticity with unknown boundaries. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 3, 1984.
4. BANICHUK N.V., Optimization of the Shape of Elastic Bodies. Nauka, Moscow, 1980.
5. LUR'E K.A., Optimal Control in Problems of Mathematical Physics. Nauka, Moscow, 1975.
6. FICHERA G., Existence Theorems in Elasticity Theory. Mir, Moscow, 1974.
7. VEKUA I.N., Principles of Tensor Analysis and Theory of Covariants. Nauka, Moscow, 1978.
8. RASHEVSKII P.K., Riemannian Geometry and Tensor Analysis. Nauka, Moscow, 1967.
9. BAKEL'MAN I.YA., VERNER A.L. and KANTOR B.E., Introduction to Differential Geometry "in the Large". Nauka, Moscow, 1973.
10. LADYZHENSKAYA O.A. and URAL'TSEVA N.N., Linear and Quasilinear Equations of Elliptic Type. Nauka, Moscow, 1973.
11. DUVAUT G. and LIONS J.-L., Inequalities in Mechanics and Physics. Nauka, Moscow, 1980.
12. MIKHAILOV V.P., Partial Differential Equations. Nauka, Moscow, 1970.

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ON CORRECT FORMULATIONS OF LEKHNITSKII PROBLEMS*

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The deformation of an elastic half-space with a cylindrical cavity under its own weight is considered. Since the solution of the problem increases at infinity, the question arises of its uniqueness and of the correct formulation of the problem itself. It is shown that two such formulations exist that yield unique solutions (that differ only to the accuracy of rigid displacements). The former corresponds to a decrease in the displacement u_j in a layer abutting on the half-space boundary, and the latter to a decrease in the stress tensor components σ_{jk}, $j, k = 1, 2$. The solutions corresponding to these formulations are distinct. They can be obtained by a passage to the limit as $D \to \infty$ from solutions of problems on the deformation of a semicylinder of diameter D with a coaxial cylindrical cavity; in the first case the side surface of the cylinder is considered rigidly clamped, and in the second stress-free.

The results are generalized to the case of non-symmetric paraboloidal cavities and elastic inclusions. Formulations are discussed of problems in which the force of gravity depends on the distance to the half-space boundary.

1. The boundary value problem and its particular solutions. Let g be a domain

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in a plane $\mathbf{R}^2$ bounded by a smooth contour ∂g and containing the origin, while G is a semi-cylinder $\{x \in \mathbf{R}^3 : y = (x_1, x_2) \in g,\ z = x_3 > 0\}$. Let Ω denote the half-space $\mathbf{R}_+^3 = \{x : x_3 > 0\}$ with the cut set G; $\Omega = \mathbf{R}_+^3 \setminus G$. Besides the Cartesian coordinates x the cylindrical coordinates (r, θ, z) will later be used, where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$.

Consider the problem of the deformation of the domain Ω due to its own weight. We denote the direction of gravity by the unit vector e. The corresponding boundary value problem has the form

$$\mu\Delta u(x) + (\lambda + \mu)\operatorname{grad}\operatorname{div} u(x) + \gamma e = 0,\quad x \in \Omega \tag{1.1}$$

$$\sigma^{(n)}(u; x) = 0,\ x \in \partial\Omega \tag{1.2}$$

where λ and μ are Lame coefficients, γ is the specific gravity, n is the external normal, $\sigma(u)$ is the stress tensor, and u is the displacement vector.

When the domain g is a circle $\{y \in \mathbf{R}^2;\ |y| < R\}$, and $e = e^{(3)}$ ($e^{(j)}$ are directions in $\mathbf{R}^3$) the problem (1.1), (1.2) is called the Lekhnitskii problem. An axisymmetric solution of this problem is presented in /1, 2/, given by the formulas

$$u_r(x) = -\nu\frac{1+\nu}{1-\nu}\frac{R^2}{Er}\gamma z,\quad u_z(x) = \frac{2\nu^2+\nu-1}{2(1-\nu)}\frac{\gamma}{E}z^2 + \nu\frac{1+\nu}{1-\nu}\gamma\frac{R^2}{E}\ln\frac{r}{R} \tag{1.3}$$

$$\sigma_{rr}(u; x) = -\frac{\nu\gamma z}{1-\nu}\left(1 - \frac{R^2}{r^2}\right),\quad \sigma_{\theta\theta}(u; x) = -\frac{\nu\gamma z}{1-\nu}\left(1 + \frac{R^2}{r^2}\right) \tag{1.4}$$
$$\sigma_{zz}(u; x) = -\gamma z$$

The components of the displacement vector and the stress tensor not indicated in (1.3) and (1.4) are equal to zero.

Moreover, in the case of axial symmetry of the domain Ω the solution of problem (1.1), (1.2) for an arbitrary vector e can be obtained as the superposition of the solution of the Lekhnitskii problem and another solution corresponding to the vector $e = e^{(1)}$. This was found by Geogdzhayev

$$\begin{aligned} u_1(x) = \frac{\gamma}{4E}\Big\{&\frac{1}{4}(3+2\nu)(1+\nu)\frac{R^4}{r^2}\cos 2\theta - \\ &\frac{r^2}{4}[(3+2\nu)(1+\nu)\cos^2\theta + (17 + 15\nu - 2\nu^2)\sin^2\theta] + \\ &(3-2\nu)(1+\nu)R^2\ln\frac{R}{r} + (1+\nu)R^2\sin^2\theta\Big\} \\ u_2(x) = \frac{\gamma}{4E}\Big\{&\frac{1}{2}(3+2\nu)(1+\nu)\frac{R^4}{r^2} + \frac{r^2}{4}(14 + 10\nu - 4\nu^2) - \\ &(1+\nu)R^2\Big\}\sin\theta\cos\theta,\ u_3(x) = -\nu(1+\nu)\frac{\gamma z}{2E}\left(\frac{R^2}{r} + r\right)\cos\theta \end{aligned} \tag{1.5}$$

$$\begin{aligned} \sigma_{rr}(u; x) &= -\frac{\gamma}{4}\left[\frac{1}{2}(3+2\nu)\left(\frac{R^4}{r^3} - r\right) + 3\left(r - \frac{R^2}{r}\right)\right]\cos\theta \\ \sigma_{\theta\theta}(u; x) &= \frac{\gamma}{4}\left[\frac{1}{4}(3+2\nu)\left(\frac{R^4}{r^3} - r\right) + (1+2\nu)r - (1-2\nu)\frac{R^2}{r}\right]\cos\theta \\ \sigma_{r\theta}(u; x) &= -\frac{\gamma}{4}\left[\frac{1}{4}(3+2\nu)\left(\frac{R^4}{r^3} - r\right) - (1-\nu)\left(r - \frac{R^2}{r}\right)\right]\sin\theta \\ \sigma_{rz}(u; x) &= \nu\gamma\frac{z}{4}\left(\frac{R^2}{r^2} - 1\right)\cos\theta \\ \sigma_{\theta z}(u; x) &= \nu\gamma\frac{z}{4}\left(\frac{R^2}{r^2} + 1\right)\sin\theta,\quad \sigma_{zz}(u; x) = 0 \end{aligned} \tag{1.6}$$

Both solutions are characterized by quadratic growth of the displacement vector components at infinity. The next three sections of this paper are devoted to constructing solutions of the homogeneous problem (1.1), (1.2) that have the same growth $O(|x|^2)$ as $|x| \to \infty$.

2. Homogeneous solutions in a half-space. At infinity the cylinder G is incident in any conical neighbourhood of the axis Ox_3, hence the principal terms of the asymptotic form of the solutions of the homogeneous problem (1.1), (1.2) as $|x| \to \infty$ should agree with the solutions of the same problem in a half-space (see /3/). We list those of the mentioned vector functions in $\mathbf{R}_+^3$ that do not decrease but have no more than quadratic growth. It is known that they are all homogeneous vector polynomials $v^{(m,q)}$ of degree $m = 0, 1, 2$, and their number is $3m$. For $m = 0$ the vectors $v^{(0,q)}$ $(q = 1, 2, 3)$ describe rigid translational displacements. Among the solutions $v^{(1,q)}$, $q = 1, \ldots, 6$, three correspond to rotations while the rest have the form

$$(x_1, -x_2, 0);\quad (x_2, x_1, 0);\quad (x_1, x_2, -2\lambda(\lambda + 2\mu)^{-1}x_3 \tag{2.1}$$

Finally, the polynomials $v^{(2,q)}$, $q = 1, \ldots, 9$ are

$$(-2x_1x_3, 0, \lambda(\lambda+2\mu)^{-1}x_3^2+x_1^2); \quad (0, -2x_2x_3, \lambda(\lambda+2\mu)^{-1}x_3^2+x_2^2) \tag{2.2}$$
$$(0, -x_3^2+x_1^2, 0); \quad (x_2^2-x_3^2, 0, 0); \quad (x_2^2-x_1^2, 2x_1x_2, 0)$$
$$(2x_1x_2, x_1^2-x_2^2, 0); \quad (x_1^2, -(4\mu+3\lambda)(\lambda+\mu)^{-1}x_1x_2,$$
$$\lambda(\lambda+\mu)^{-1}x_1x_3)$$
$$(-(4\mu+3\lambda)(\lambda+\mu)^{-1}x_1x_2, x_2^2, \lambda(\lambda+\mu)^{-1}x_2x_3);$$
$$(x_2x_3, x_1x_3, -x_2x_1)$$

The stresses $\sigma_{jk}^{(m,q)} = \sigma_{jk}(v^{(m,q)})$ corresponding to the displacements (2.1), (2.2) are given by the equations

$$\sigma_{11}^{(1,1)} = -\sigma_{22}^{(1,1)} = \sigma_{12}^{(1,2)} = 2\mu; \quad \sigma_{11}^{(1,3)} = \sigma_{22}^{(1,3)} = 2\mu(2\mu+3\lambda)(2\mu+\lambda)^{-1} \tag{2.3}$$

$$\sigma_{22}^{(2,2)} = \sigma_{11}^{(2,1)} = -8\mu(\lambda+\mu)(\lambda+2\mu)^{-1}x_3; \tag{2.4}$$
$$\sigma_{22}^{(2,1)} = \sigma_{11}^{(2,2)} = -4\mu\lambda(\lambda+2\mu)^{-1}x_3$$
$$\sigma_{12}^{(2,2+j)} = 2\mu x_j; \quad \sigma_{3,3-j}^{(2,2+j)} = -2\mu x_3; \quad \sigma_{jj}^{(2,4+j)} = -4\mu x_j;$$
$$\sigma_{3-j,3-j}^{(2,4+j)} = 4\mu x_j; \ \sigma_{12}^{(2,4+j)} = 4\mu x_{3-j}; \quad \sigma_{jj}^{(2,6+j)} = 2\mu(\lambda+2\mu)(\lambda+\mu)^{-1}x_2$$
$$\sigma_{3-j,3-j}^{(2,6+j)} = -8\mu x_2; \quad \sigma_{3j}^{(2,6+j)} = \lambda\mu(\lambda+\mu)^{-1}x_3$$
$$\sigma_{12}^{(2,6+j)} = -\mu(3\lambda+4\mu)(\lambda+\mu)^{-1}x_{3-j}; \quad \sigma_{12}^{(2,9)} = 2\mu x_3, \quad j = 1, 2$$

The stress tensor components not presented in (2.3), (2.4) are zero.

3. The problem on a plane with an orifice. The solution of the homogeneous problem in a half-space leaves a residual in the boundary conditions on ∂G. This residual can be cancelled by solutions of the boundary value problem in $\mathbf{R}^2 \setminus g$. The appropriate differential operators are obtained from the operators in system (1.1), (1.2) by elimination of the derivative with respect to z. Therefore, a set of the plane problem of the theory of elasticity and the problem of antiplane shear in $\mathbf{R}^2 \setminus g$ arises

$$\mu\nabla\cdot\nabla W(y) + (\lambda+\mu)\nabla\nabla\cdot W(y) = 0, \quad \mu\nabla\cdot\nabla w_3(y) = 0, \quad y \in \mathbf{R}^2 \setminus g \tag{3.1}$$

$$\Sigma^{(n)}(W; y) = P(y), \quad \mu\frac{\partial w_3}{\partial n}(y) = p_3(y), \quad y \in \partial g \tag{3.2}$$

Here $W = (w_1; \ w_2)$; $\nabla = \nabla_y$; Σ is a two-dimensional stress tensor, and n is the unit external normal to ∂g. We denote the three-dimensional vector (w_1, w_2, w_3) by w. The solution of problem (3.1), (3.2) exists for any smooth loads $P = (p_1, p_2)$, p_3 and allows of the asymptotic representation

$$w(y) = \sum_{j=1}^{3}\left\{c_j T^{(j)}(y) + \sum_{k=1}^{2} c_{jk}\frac{\partial}{\partial y_k}T^{(j)}(y)\right\} + O(|y|^{-2}) \tag{3.3}$$

(see /4/, for example), where c_j, c_{jk} are constants, $T^{(j)}$ are columns of a block matrix consisting of a two-dimensional Somigliani tensor (the elements in the first two rows and columns) and the fundamental solution $(2\pi\mu)^{-1}\ \ln|y|^{-1}$ of the operator $-\mu\nabla\cdot\nabla$ in $\mathbf{R}^3$ (the element in the lower right-hand corner).

The solution (3.3) has a logarithmic increase at infinity. It decreases only if the principal load vector P and the mean p_3 on ∂g equal zero (then $c_j = 0$). If

$$p_j(y) = \sum_{k=1}^{3} n_k(y)(a_0^{(j,k)} + a_1^{(j,k)}y_1 + a_2^{(j,k)}y_2) \tag{3.4}$$

in (3.2) and $a_p^{(j,k)}$ are constants, then by using the method of /5/, expressions can be obtained for the coefficients c_j in (3.3)

$$c_j = -|g|\,(a_1^{(j,1)} + a_2^{(j,2)}) \tag{3.5}$$

Here $|g|$ is the area of the domain g.

4. Solutions of the homogeneous problem in Ω. We construct solutions of the homogeneous problem (1.1), (1.2) that do not decrease at infinity. Six are trivial, rigid displacements. The rest have mainly the asymptotic form (2.1) or (2.2). We will first examine the displacement (2.1). According to (2.3), the vectors $v^{(1,q)}$ $(q = 1, 2, 3)$ leave a residual of the form (3.4) in the boundary conditions (1.2) on ∂G, $a_1^{(j,k)} = a_2^{(j,k)} = 0$. By virtue of (3.3) and (3.5), solutions $w^{(1,q)}(y)$ exist that decrease as $O(|y|^{-1})$ as $|y| \to \infty$ and cancel these residuals. We set

$$V^{(1,q)}(x) = v^{(1,q)}(x) + \chi(z^{-1}|y|)\,w^{(1,q)}(y) \tag{4.1}$$

where χ is a shearing function from $\mathbf{C}^\infty(\mathbf{R}_1)$; $\chi(t) = 1$ for $t \leqslant 2R$ and $\chi(t) = 0$ for $t \geqslant 3R$;

the number R is selected so that the domain g is incident in a circle of radius R.

We consider the residual of the vector (4.1) in the homogeneous Eqs.(1.1), (1.2). We are here interested only in its behaviour as $|x| \to \infty$. In this sense, the boundary conditions are satisfied completely and the residual $F^{(1,q)}$ in the Lamé system is concentrated in the cone $K_R = \{x : z^{-1}|y| \in (2R, 3R)\}$. The inequalities

$$c_1 z < |x| < C_1 z, \quad c_2|y| < |x| < C_2|y| \tag{4.2}$$

are valid for points of this cone, where c_k and C_k are certain positive constants. Consequently we finally obtain the estimate $|F^{(1,q)}(x)| \leqslant \text{const}\,|x|^{-3}$ for $F^{(1,q)}$. Therefore, there exists an energetic solution $u^{(1,q)}$ of the problem of the theory of elasticity in Ω, that vanishes at infinity and cancels the mentioned residual while the sum

$$U^{(1,q)}(x) = V^{(1,q)}(x) + u^{(1,q)}(x), \quad q = 1, 2, 3 \tag{4.3}$$

is a solution of the homogeneous problem (1.1), (1.2).

We will now consider the displacement (2.2). Because of (2.4), the residual of the vector $V^{(2,q)}$ in the boundary conditions (1.2) on ∂G has the form (3.4) in which $a_1^{(j,k)}$, $a_2^{(j,k)}$ are constants, while $a_0^{(j,k)} = c_{jk}z$, $c_{jk} = \text{const}$. Hence, a vector function $w^{(2,q)}(y, z) = w^{(2,q,0)}(y) + zw^{(2,q,1)}(y)$, exists subject to the homogeneous Lamé system and cancelling the residual in the boundary conditions on ∂G such that

$$w^{(2,q,1)}(y) = O(|y|^{-1}), \quad w^{(2,q,0)}(y) = O(|\ln|y||)$$

The vector $v^{(2,q)}(x) + \chi(z^{-1}|y|)w^{(2,q)}(y)$ satisfies the boundary conditions on $\partial\Omega$ outside a certain sphere and the residual in the Lamé system is concentrated in the cone K_R and allows of the representation

$$F^{(2,q)}(x) + O(|x|^{-3}|\ln|x||) \tag{4.4}$$

(here the inequalities (4.2) are used in estimating the remainder). The first component in (4.4) has the form

$$F^{(2,q)}(x) = |x|^{-2}(\ln|x|f^{(q,0)}(x|x|^{-1}) + f^{(q,1)}(x|x|^{-1})) \tag{4.5}$$

where $f^{(q,j)}$, $j = 0, 1$ are smooth functions on the hemisphere $S_+^2 = \{x : |x| = 1, x_3 > 0\}$ (their carriers are in a ring cut out by the cone K_R on S_+^2).

As is shown (in a more general situation) in /4, 5/, the Lamê system with right-hand side (4.5) in $\mathbf{R}_+^3$ under the condition that the half-space boundary in stress-free, has the particular solution

$$\psi^{(2,q)}(x) = (\ln|x|)^2\,\psi^{(2,q,2)}\left(\frac{x}{|x|}\right) + \ln|x|\,\psi^{(2,q,1)}\left(\frac{x}{|x|}\right) + \psi^{(2,q,0)}\left(\frac{x}{|x|}\right) \tag{4.6}$$

($\psi^{(2,q,n)}$ are smooth functions in S_+^2).

The solution (4.6) cancels the principal part (4.5) of the residual (4.4) in the Lame system, but it leaves a residual P, in its turn, in the boundary condition on ∂G. It can be confirmed that

$$P(x) = z^{-1}[(\ln z)^2 p^{(q,2)}(y) + \ln z\,p^{(q,1)}(y) + p^{(q,0)}(y)] + O(z^{-2}|\ln z|^2) \tag{4.7}$$

where the components of the vectors $p^{(q,n)}$ possess a zero mean on ∂g. According to Sect.3 solutions $\Psi^{(q,n)}$ of the problem (1.1), (1.2) with right-hand sides $p^{(q,n)}$ exist such that $\Psi^{(q,n)} = O(|y|^{-1})$.

We finally set

$$\begin{aligned} V^{(2,q)}(x) &= v^{(2,q)}(x) + \chi(z^{-1}|y|)\,w^{(2,q)}(y, z) + \psi^{(2,q)}(x) + \\ &\chi(z^{-1}|y|)\,z^{-1}[(\ln z)^2\,\Psi^{(q,2)}(y) + \ln z\,\Psi^{(q,1)}(y) + \Psi^{(q,0)}(y)] \end{aligned} \tag{4.8}$$

Taking account of (4.5), (4.7), we obtain that the vector-function (4.8) leaves the residuals F and S in the homogeneous Lamé system and in the boundary condition (1.2), respectively, for which the following estimates hold:

$$\begin{aligned} &|S(x)| \leqslant \text{const}\,(|x|^{-2}|\ln|x||^2) \\ &|F(x)| \leqslant \text{const}\,|x|^{-3}\,|\ln|x|| \quad \text{when} \quad |y| > 3Rz \\ &|F(x)| \leqslant \text{const}\,|x|^{-2}\,|\ln|x||\,(|x|^{-1} + |y|^{-2}|\ln|x||) \\ &\text{when} \quad |y| < 3Rz \end{aligned}$$

It therefore follows that the following integrals are finite

$$\int_\Omega |x|^2|F(x)|^2\,dx, \quad \int_{\partial\Omega} |x|^2|S(x)|^2\,dx$$

and, therefore, energetic solutions $u^{(2,q)}$ exist that vanish at infinity and also cancel these residuals. Hence, the vector-functions

$$U^{(2,q)}(x) = V^{(2,q)}(x) + u^{(2,q)}(x), \quad q = 1, \ldots, 9 \tag{4.9}$$

are solutions of the homogeneous problem (1.1), (1.2). Moreover, all the displacements (4.3) and (4.9) are linearly independent and the energy integral becomes infinite. Confirmation of the fact that all the solutions U of the homogeneous problem (1.1), (1.2) subject to the inequality

$$|U(x)| \leqslant \text{const}\,|x|^2 \tag{4.10}$$

are exhausted by linear combinations of the rigid displacements and the vector-functions (4.3), (4.9), requires reliance upon complex and awkward mathematical apparatus using the results and general methods of investigating elliptical boundary value problems in domains with singular points (see /3—8/). The appropriate proof is omitted in this paper.

5. Correct formulations of the Lekhnitskii problem. We will now consider axisymmetric problem whose particular solution was found by Lekhnitskii and which has the form of (1.3), (1.4). Among the linear combinations of vectors (2.1), (2.2), there is just one vector that is invariant under rotation around the Oz axis. It equals (apart from a factor)

$$v(x) = v^{(2,1)}(x) + v^{(2,2)}(x) = (-2y_1z;\ -2y_2z;\ r^2 + 2\lambda(\lambda+2\mu)^{-1}z^2) \tag{5.1}$$

The solution $U(x) = U^{(2,1)}(x) + U^{(2,2)}(x)$ of the homogeneous problem (1.1), (1.2)

$$\begin{gathered} U_r(r,z) = -2zr + 2z\frac{1+\nu}{1-\nu}\frac{R^2}{r} \\ U_z(r,z) = \frac{2\nu z^2}{1-\nu} + r^2 + 2\frac{1+\nu}{1-\nu}R^2\ln\frac{r}{R}, \quad U_\theta(r,z) = 0 \end{gathered} \tag{5.2}$$

corresponds to this vector.

This solution is constructed as shown in Sect.4. The circumstance that the algorithm proposed yields the solution in closed form is accidental; in general, it will enable us to find just the asymptotic form. The stress tensor components $\sigma(U)$ are defined by the relationships

$$\begin{gathered} \sigma_{rr}(U;\ r,\ z) = -2\mu z\frac{1+\nu}{1-\nu}\left(1-\frac{R^2}{r^2}\right), \quad \sigma_{\theta\theta}(U;\ r,\ z) = \\ -2\mu z\frac{1+\nu}{1-\nu}\left(1+\frac{R^2}{r^2}\right), \quad \sigma_{zz}(U;\ r,\ z) = \sigma_{rz}(U;\ r,\ z) = 0 \end{gathered} \tag{5.3}$$

Comparing (5.2) and (1.3) we see that a decrease in the components u_1 and u_2 as $|x| \to \infty$ is characteristic for the displacements (1.3) in the layer $Q_d = \{x \in \Omega : z < d\}$ of arbitrary thickness d, while the displacements (5.2) do not possess this property. Consequently, the additional conditions that yield the uniqueness theorem for the solution (1.3), (1.4) of the Lekhnitskii problem have the form

$$u_j(x) = o(1) \quad \text{as} \quad |x| \to \infty,\ x \in Q_d;\quad j = 1, 2 \tag{5.4}$$

It is clear that in order to satisfy (5.4) it is necessary to assume that the constant C in the linear combination

$$u^{(1)} = u + CU \tag{5.5}$$

equals zero.

On the other hand, the stresses σ_{rr} and $\sigma_{\theta\theta}$ in (5.3) and (1.4) have identical behaviour at infinity, apart from a factor. Consequently, the validity of the following constraints

$$\sigma_{jk}(u^{(1)};\ x) = o(1) \quad \text{as} \quad |x| \to \infty, \quad x \in Q_d;\ j, k = 1, 2 \tag{5.6}$$

can be achieved for the vector (5.5).

Direct calculations show that for $c = -\nu\gamma\,[2\mu(1+\nu)]^{-1}$ the vector (5.5) is given by the formulas

$$u_r^{(1)}(x) = \nu E^{-1}\gamma zr, \quad u_z^{(1)}(x) = -\gamma(2E)^{-1}(z^2 + \nu r^2), \quad u_\theta^{(1)}(x) = 0 \tag{5.7}$$

while the corresponding stresses have the form

$$\begin{gathered} \sigma_{rr}(u^{(1)};\ x) = \sigma_{\theta\theta}(u^{(1)};\ x) = \sigma_{rz}(u^{(1)};\ x) = 0, \\ \sigma_{zz}(u^{(1)};\ x) = -\gamma z \end{gathered} \tag{5.8}$$

Thus the Lekhnitskii problem allows of two formulations in the class of vector-functions subject to the relationship (4.10). The first is associated with the additional requirement (5.4) of a decrease in the displacement component u_r in any layer of finite thickness, and the second with the requirement (5.6) of a decrease in the stress tensor components σ_{rr}, $\sigma_{\theta\theta}$.

The unique solutions corresponding to these formulations are given by (1.3), (1.4) and (5.7), (5.8).

6. Correct formulations of the problem of a vertical half-space with a cylindrical cavity. It is seen from (1.5) that the solution of problem (1.1), (1.2) constructed by Geogdzhayev for $e = e^{(1)}$ has a quadratic increase $O(|x|^2)$ in the layer Q_d abutting on the half-space boundary.

As is shown in Sect.4, the solution of this problem is determined to the accuracy of a linear combination of the twelve vector-functions (4.3), (4.9). We select coefficients of this linear combination such that its sum with the solution (1.5) has the least possible growth in the layer Q_d. Omitting the computations, we write down the result

$$u^{(2)} = u + \gamma(1+\nu)[16E]^{-1}\{16U^{(2,4)} + (1-2\nu)U^{(2,5)} + 4U^{(2,7)}\} \tag{6.1}$$

By using the procedure elucidated in Sect.4, we construct the asymptotic form of the solution (6.1) of problem (1.1), (1.2), $e = e^{(1)}$. We select the solution of the Lamé system (1.1) in the form

$$v(x) = -\gamma(2\mu)^{-1}(z^2, 0, 0) \tag{6.2}$$

The stresses $\sigma_{rr}(v)$ and $\sigma_{r\theta}(v)$ equal zero identically, while $\sigma_{rz}(v; x) = -\gamma z\cos\theta$. Consequently, the vector

$$zw^{(1)}(y) + w^{(0)}(y) \tag{6.3}$$

is the next approximation to $u^{(1)}$.

The component $w_3^{(1)}$ of the vector-function $w^{(1)} = (0, 0, w_3^{(1)})$ from (6.3) is a solution of the external Neumann problem

$$\mu\Delta w_3^{(1)}(y) = 0, \quad y \in \mathbf{R}^2 \setminus g, \quad \mu(\partial w_3^{(1)}/\partial r)(y) = \gamma\cos\theta, \quad y \in \partial g$$

(see Sect.4) and is given by the equality

$$w^{(1)}(y) = (0, 0, -\gamma R^2(\mu r)^{-1}\cos\theta) \tag{6.4}$$

(We recall that the domain g is a circle of radius R). The vector $W^{(0)}$ comprised of the first two components of the vector $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, 0)$ in (6.3) is a solution of the plane problem of the theory of elasticity

$$\mu\nabla\cdot\nabla W^{(0)}(y) + (\lambda+\mu)\nabla\nabla\cdot W^{(0)}(y) - \gamma R^2(\lambda+\mu)\mu^{-1}\nabla\times(r^{-1}\cos\theta) = 0, \ y \in \mathbf{R}^2 \setminus g$$

$$\Sigma_{rr}(W^{(0)}; R, \theta) = \gamma R\lambda\mu^{-1}\cos\theta, \quad \Sigma_{r\theta}(W^{(0)}; R, \theta) = 0$$

and is calculated by means of the formula

$$W^{(0)}(r, \theta) = \frac{\gamma R^2}{2\mu\nu}\begin{Vmatrix}(3-2\nu)\ln(rR^{-1})\cos\theta \\ -[(3-2\nu)\ln(rR^{-1})+1]\sin\theta\end{Vmatrix} - \frac{\gamma R^4(1+2\nu)}{16\mu r^2}\begin{Vmatrix}\cos\theta \\ \sin\theta\end{Vmatrix} - \frac{\gamma R^2(4-\nu)(3-4\nu)}{8\mu(1-\nu)}\begin{Vmatrix}\ln(rR^{-1})\cos\theta \\ -[\ln(rR^{-1})+(3-4\nu)^{-1}]\sin\theta\end{Vmatrix} \tag{6.5}$$

Thus, the approximation has the form

$$V(x) = v(x) + zw^{(1)}(y) + w^{(0)}(y) \tag{6.6}$$

(Here, compared with (4.1) and (4.8), there is no multiplication by the shear since the vector-functions (6.4) and (6.5) are defined everywhere in Ω). In order for the sum (6.6) to satisfy the homogeneous boundary conditions (1.2) on the half-space boundary, satisfaction of the relationship

$$\sigma_{33}(zw^{(1)}(y) + w^{(0)}(y)) \equiv (2\mu+\lambda)w_3^{(1)}(y) + \nabla\cdot W^{(0)}(y) = 0 \tag{6.7}$$

is necessary.

By direct verification from (6.4) and (6.5) we deduce that (6.7) does not hold. Therefore, the displacements (6.6) are only asymptotic

$$u^{(2)}(x) = V(x) + O(|\ln|x||^2) \tag{6.8}$$

and not the exact value of the solution. Such a solution would occur in both versions of the Lekhnitskii problem in Sect.5 (it was mentioned there that this circumstance is accidental). Consequently, solution (6.1) is not defined explicitly in terms of elementary functions and we can only speak of its existence.

The solution mentioned is characterized by slight growth of the displacement in the layer Q_d (see (6.8) and (6.2)-(6.6)). Since all the solutions (4.3), (4.9) of the homogeneous problem grow as $O(r)$ or $O(r^2)$, in this layer according to (2.1), (2.2), then the solution subject to the condition

$$u^{(2)}(x) = O(|\ln|x||^2) \quad \text{as} \quad |x| \to \infty,\ x \in Q_d \tag{6.9}$$

is unique (apart from rigid displacements).

We will now examine the sum of the solution (1.5) and a linear combination of the vector-functions (4.3), (4.9). The stresses (1.6) grow as $O(r^2)$ in Q_d. We find the coefficients of the mentioned linear combination so as to guarantee the least possible growth of the stress in this layer. Omitting simple but cumbersome computations, we present the result

$$\begin{aligned} &u^{(3)} = u + c_1 v^{(2,4)} + c_2 v^{(2,5)} + c_3 v^{(2,7)} \\ &(c_1 = -2c(2\nu^2 + 9\nu + 2),\quad c_2 = c(2\nu^2 + 9\nu - 13), \\ &c_3 = -10c,\ c = \gamma(16E)^{-1}) \end{aligned} \tag{6.10}$$

All the stresses constructed by means of the displacement (6.10) decrease in the layer Q_d. Only the component $\sigma_{13}(u^{(3)})$, which equals $\gamma z/4 + o(1)$ and is only bounded is the exception.

As formulas (2.3), (2.4) show, the functions $\sigma_{jk}(U^{(n,q)})$, where $n = 1, 2$, $q = 1, \ldots, 3^n$, $j, k = 1, 2$, do not vanish at infinity. Consequently, the solution (6.10) of problem (1.1), (1.2) is unique for $e = e^{(1)}$ apart from rigid displacements if it is subject to the additional condition (5.6).

Thus, (6.1) and (6.10) are two distinct solutions of problem (1.1), (1.2) for $e = e^{(1)}$ that correspond to two different formulations: satisfaction of condition (6.9) or condition (5.6).

7. A problem with an arbitrary direction of gravity. As already mentioned, the solution of problem (1.1), (1.2) is obtained in the case of an arbitrary vector e by the superposition of solutions of the Lekhnitskii problem and of the problem examined in Sect.6. We formulate results following from the assertions of Sects.5 and 6. Despite the fact that the case of a circular cylinder was considered in these sections, the procedure described in Sect.4 enables us to conclude that all that was said about the corresponding problems is conserved even in the case of an arbitrary section g. Consequently, we shall henceforth speak of the non-symmetric problem.

The addition of condition (6.9) yields the first correct formulation of the problem. The solution of problem (1.1), (1.2), (6.9) is unique apart from rigid translational displacements. The constraint (6.9) can here be weakened and replaced by the following

$$u(x) = o(|x|) \quad \text{as} \quad |x| \to \infty,\ x \in Q_d \tag{7.1}$$

since, as before, the homogeneous solutions (4.3) and (4.9) do not satisfy relationship (7.1).

The second correct formulation is ensured by fixing the behaviour of the stresses in the layer Q_d. The appropriate problem (1.1), (1.2), (5.6) has a unique solution (apart from rigid displacements). Note that the condition (5.6) can be strengthened by imposing constraints on the vector $\sigma^{(3)}(u)$:

$$\sigma_{jk}(u; x) = o(1),\ \sigma^{(3)}(u; x) = O(1) \quad \text{as} \quad |x| \to \infty,\ x \in Q_d \tag{7.2}$$

Problem (1.1), (1.2), (7.2) is equivalent to problem (1.1), (1.2), (5.6).

8. Generalizations and corollaries. 1°. We note that both solutions (1.3), (1.4) and (5.7), (5.8) corresponding to the correct formulations of the Lekhnitskii problem presented in Sect.5 can be obtained from the solution of the problem in a semicylinder of large diameter D, by passing to the limit as $D \to \infty$. Namely, let $C_D = \{x \in \mathbf{R}^3 : z > 0,\ r < D/2\}$, $\Gamma_D = \{x \in \mathbf{R}^3;\ z > 0,\ r = D/2\}$ and $\Omega_D = \Omega \cap C$. Then the solution (1.3), (1.4) of problem (1.1), (1.2), (5.4) for $e = e^{(3)}$ is the limit of the solution u_D of the problem in Ω_D with rigid clamping conditions on Γ_D as $D \to \infty$. The solution (5.7), (5.8) of problem (1.1), (1.2), (5.6) for $e = e^{(3)}$ is, in turn, the limit of the solutions of such problems under the condition that the side surface Γ_D is stress-free.

2°. The method presented in Sect.4 for constructing solutions of the homogeneous problem has a broader domain of application. Any "paraboloid"

$$\Pi = \{x \in \mathbf{R}^3 : h(z)^{-1} y \in g,\ z > 0\};\ h(z) = z^{1-\delta} H(z^{-1}),\ \delta > 0 \tag{8.1}$$

where $t \to H(t)$ is a function smooth in the neighbourhood of the point $t = 0$, can be proposed as the set G introduced in Sect.1. The domain Ω can here be formed by the removal of $\bar{G}$ from any non-empty open cone K with smooth directrix containing the Oz axis. The domains G and Ω should coincide with Π and $K \setminus \bar{\Pi}$ only outside some sphere; the behaviour of the boundary at a finite distance is not essential.

3°. Moreover, it can be considered that the body G is filled with a material with other Lamé constants λ_0 and μ_0, i.e., the conjugate problem can be considered:

$$\begin{aligned}&\mu\Delta u(x)+(\lambda+\mu)\operatorname{grad}\operatorname{div}u(x)+\gamma e=0,\quad x\in\Omega\\&\mu_0\Delta u^\circ(x)+(\lambda_0+\mu_0)\operatorname{grad}\operatorname{div}u^\circ(x)+\gamma_0 e=0,\quad x\in G\\&\sigma^{(n)}(u;x)=0,\quad x\in\partial\Omega\setminus\partial G;\quad \sigma_0^{(n)}(u^\circ;x)=0,\quad x\in\partial G\setminus\partial\Omega\\&u(x)=u^\circ(x),\quad \sigma^{(n)}(u;x)=\sigma_0^{(n)}(u^\circ;x),\ x\in\partial G\cap\partial\Omega\end{aligned}\tag{8.2}$$

All the quantities referring to the inclusion are provided with the superscript $^\circ$. An analogous problem for a plane with limited inclusion g is here obtained as a "model" problem from Sect.3. The properties of the solutions of such a problem are in no way different from those elucidated in Sect.3. The procedure for constructing the asymptotic form from Sect.4 is thereby carried over also to the case of problem (8.2).

Attention should be given to the fact that the passage to the case of an absolutely rigid inclusion within the framework of the mathematical results presented is impossible in problem (8.2). The reason for this is that the properties of the solutions of the corresponding model problem (system (3.1) with the boundary conditions $W=\Phi$, $w_3=\varphi_3$ on ∂g) differ radically from the properties of problem (3.1), (3.2).

4°. We note that the proceudre presented in Sect.4 for seeking the solutions of the homogeneous problem is similar to the algorithm /9, 10/ for constructing the asymptotic form of solutions of elliptic boundary value problems with small singular perturbations of the domain.

5°. The two formulations presented in Sect.7 for problem (1.1), (1.2), within whose framework the uniqueness theorem holds, hold even in the case of problem (8.2). The domain G can here be not only a semicylinder with section g (the case $\delta=1$ in (8.1)), but also a set shrinking at infinity (the case $\delta>1$) and a "paraboloidal" set expanding at infinity (the case $\delta\in(0,1)$).

6°. Let us examine the situation when gravity acts in the direction of the Oz axis but varies as the "depth" increases, i.e., the mass forces f in system (1.1) equal $\gamma(z)e^{(3)}$. We confine ourselves here to a study of the cylindrical domain G (with the arbitrary section g).

We assume that the function γ decreases as $o(z^{-1-\delta})$, $\delta>0$, at infinity. We set

$$u(x)=v(x)+e^{(3)}\varphi(z),\quad \varphi(z)=\frac{1}{2\mu+\lambda}\int_0^z\int_0^t\gamma(\tau)\,d\tau\,dt\tag{8.3}$$

Then $\varphi(z)=bz+O((1+z)^{1-\delta})$, where $b=\text{const}$ and the vector-function v is a solution of the boundary value problem

$$\begin{aligned}&\mu\Delta v(x)+(\lambda+\mu)\operatorname{grad}\operatorname{div}v(x)=0,\quad x\in\Omega\\&\sigma^{(2)}(v;x)=0,\quad x\in\partial\Omega\setminus\partial G;\quad \sigma^{(n)}(v;x)=-\lambda\varphi'(z)\,n(x),\\&x\in\partial G\cap\partial\Omega\end{aligned}\tag{8.4}$$

The principal term w of the asymptotic form v has the form $w=(W,0)$, where W is a solution of problem (3.1), (3.2) that vanishes at infinity for $P=-\lambda\varphi'(n_1,n_2)$, $p_3=0$. It leaves residuals F and S in the Lamé system and the boundary condition on $\partial\Omega\setminus\partial G$; $|F(x)|=O(r^{-2}(1+z)^{-1-\delta})$, $|S(x)|=O(r^{-1})$. As in Sect.4, a solution exists that cancels these residuals and reaches the order of magnitude $O(|\ln|x||)$. Therefore, by repeating the discussions from Sects.5 and 7, we obtain that the problem under consideration is uniquely solvable in the class of vector-functions subject to the relationships (7.1).

In order to obtain the solution of the problem in the case of the constraint (5.6), we should set

$$u(x)=v(x)+[4\mu(2\mu+3\lambda)]^{-1}(-2\lambda r\varphi'(z),0,4(\mu+\lambda)\varphi(z)+\varphi''(z)r^2\lambda)$$

Problem (8.4) also undergoes corresponding changes.

Finally, we assume that the mass forces in (1.1), (1.2) are determined by the formula

$$f(x)=\gamma(r^\delta z)e^{(3)}$$

where γ is a smooth function $\delta>0$. In order for an energetic solution to exist (this solution is unique), the inclusion $|x|f\in L_2(\Omega)$ is necessary, or equivalently

$$\int_\Omega|x|^2|f(x)|^2dx=\int_0^\infty|\gamma(t)|^2\int_{\mathbf{R}^2\setminus g}r^{-\delta}(r^2+r^{-2\delta}t^2)\,dy<\infty$$

This latter inequality is satisfied only if $(1+t)\gamma(t)\in L_2(0,+\infty)$ and $\delta>4$.

REFERENCES

1. LEKHNITSKII S.G., Symmetric deformation and torsion of an anisotropic solid of revolution with anisotropies of particular form, PMM, 4, 3, 1940.
2. LEKHNITSKII S.G., Theory of Elasticity of an Anisotropic Body. Nauka, Moscow, 1977.

3. MAZ'YA V.G., NAZAROV S.A. and PLAMENEVSKII B.A., Elliptical boundary value problems in domains of the type of the exterior of a peak, Problems of Mathematical Analysis, 9, Izd. Leningrad Univ. Leningrad, 1984.
4. KONDRAT'EV V.A., Boundary value problems for elliptic equations in domains with conical or angular points, Trudy Moscow Math. Soc., 16, 1967.
5. MAZ'YA V.G. and PLAMENEVSKII B.A., On coefficients in the asymptotic of solutions of elliptic boundary value problems in domains with conical points. Math. Nachr., 76, 1977.
6. MAZ'YA V.G. and PLAMENEVSKII B.A., Etimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for the solutions of elliptic boundary value problems in domains with singular points on the boundary. Math. Nachr. 81, 1978.
7. PLAMENSEVSKII B.A., On the asymtotic behaviour of the solutions of quasi-elliptical differential equations with operator coefficients. Izv. Akad. Nauk SSSR, Ser. Matem., 37, 6, 1973.
8. KONDRAT'EV V.A. and OLEINIK O.A., Boundary value problems for partial differential equations in non-smooth domains. Usp. Matem. Nauk, 38, 2, 1983.
9. MA'YA V.G., NAZAROV S.A. and PLAMENEVSKII B.A., Asymptotic form of the Solution of Elliptic Boundary Value Problems for Singular Perturbed Domains. Izd. Tbil. Univ., Tbilisi, 1981.
10. NAZAROV S.A., Introduction to Asymptotic Methods in the Theory of Elasticity. Izd. Leningrad Univ. Leningrad, 1983.

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ELASTIC EQUILIBRIUM OF A PLATE WITH A PARTIALLY REINFORCED CURVILINEAR HOLE*

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An approximate method is proposed to determine the state of stress near a curvilinear hole whose outline is partially reinforced by a thin elastic rod of variable cross-section in an infinite plate. The problem is reduced to a system of two singular integral equations in the contact stresses by the method of complex-function theory /1/, and the method of boundary collocation is used to solve them /2/. Certain special cases of the problem and numerical examples are examined.

Problems of reducing the stress concentration around circular holes in plates have been discussed in /3, 4/. In practice, all the previous investigations on this problem have been devoted to problems of reinforcement of the whole hole outline by rods of constant or variable cross-section.

1. We consider an infinite isotropic plate of thickness $2h$ with a circular hole of radius $\rho_0 = 1$. Part of the hole outline, determining the central angle $2\alpha_0$, is reinforced by a thin elastic rod of variable cross-section. We consider the thickness of the reinforcement to be constant, and the width to be a continuous smooth function of the arc. The plate is subjected to bending in two mutually perpendicular directions by the moments $M_1 = M_x^\infty$, $M_2 = M_y^\infty$ applied at "infinity". There is not external load on the hole outline.

The plate middle plane is referred to a ρ, λ polar coordinate system with the pole at the centre of the hole. The polar axis passes through the middle of the reinforcing rod and makes an angle β_0 with the direction of action of the moment M_1. We consider the rod as an elastic line subjected to bending and torsion /3/.

The boundary conditions of the boundary value problem and its solution on the hole outline, in the notation of /1/, have the form

$$\varkappa\Phi^-(t_0) + \Phi^+(t_0) = -kf(t_0) \tag{1.1}$$
$$\Phi^+(t_0) - \Phi^-(t_0) = k\,(1+\nu)^{-1}\,[M_\lambda - \nu M_\rho - i\,(1+\nu)\,H_{\rho\lambda}]$$

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